3D Origami Design based on Tucking Molecule

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Abstract

This paper presents a new approach for folding a piece of paper into an arbitrary three-dimensional polyhedral surface based on "tucking molecules." The approach is to align the polygons of the three-dimensional surface onto a convex region of a plane and fill the blank area between the polygons with "tucking molecules." Tucking molecules are aligned between segments and vertices of the polygons, and "paste" them together by being folded flat. We propose a novel method for generating crease pattern for tucking molecules. Necessary and sufficient conditions to enable construction of the required surface are also shown. Additionally we show that the necessary conditions can be represented in inequality conditions by using the "tuck proxies."

1 Introduction and Previous Methods

Designing arbitrary three-dimensional surface by origami is one of the ultimate objectives of origami designing. There have been many approaches for designing three-dimensional models.

Most popular way of designing a three-dimensional origami model is by shaping a flat-folded uniaxial base with controlled lengths of flaps. Methods for designing uniaxial base is mainly investigated by Toshiyuki Meguro [1], Fumiaki Kawahata [2], and Robert Lang [3], [4]. This method is good for organic models such as insects and some animals. However it is not possible to specifically control the three-dimensional shape because the only controllable parameters when designing a base are lengths of the flaps.

Many origami artists are also engaged in two-dimensional design of origami bases. A design method for 2D origami base based on placing facets are recently investigated by Toshiyuki Meguro [5]. For designing the silhouette of an arbitrary two-dimensional polygon, a proof and a method is shown by Demaine et.al. [6], and another method is proposed by Masahiko Tanaka [7].

Demaine et.al. [6] proved that any polyhedron can be folded from a sheet of paper. The method used for the proof is based on wrapping a strip of paper around a polyhedron and it is practically not possible to design real models by the method.

Some arbitrary polyhedral models have been designed. Both roofs by the author [8] and a frog by Masahiko Tanaka [7] use symmetry of tucks which constrains the alignment of the polygons on paper.

In this paper, we propose a new method for designing an arbitrary polyhedral surface (Figure 1). Instead of using the symmetry of tucks between edges, we propose "edge-tucking molecules," which can be generated for a given alignment of edges. We further

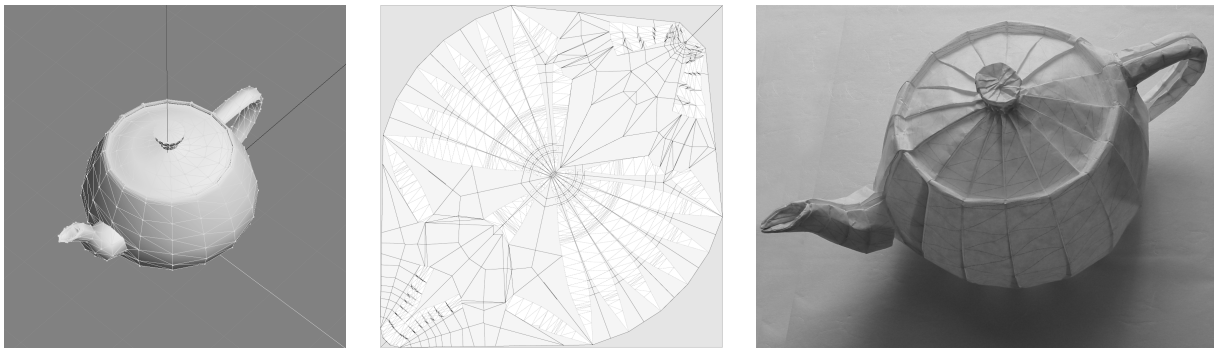


Figure 1: A 3D teapot can be folded (right) from the crease pattern (middle) constructed from a teapot represented by a triangle mesh (left)

investigate the conditions for three-dimensional curvature, which is the most distinctive part of 3D origami design.

2 Method Overview

2.1 Tucking Molecules

Our approach is to align polygons of the polyhedral surface onto a convex region of a plane and fill the area between polygons with "edge-tucking molecules" and "vertex-tucking molecules." Figure 2 shows how the paper is tessellated into polygons and molecules.

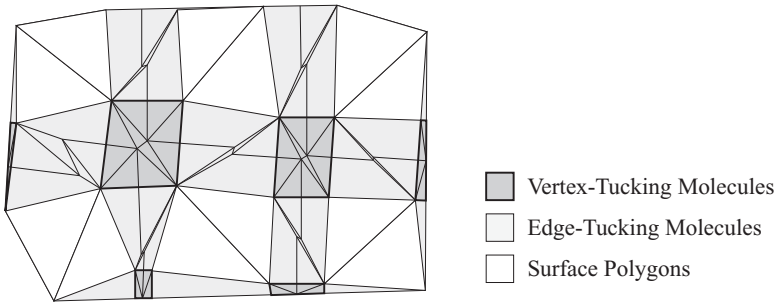
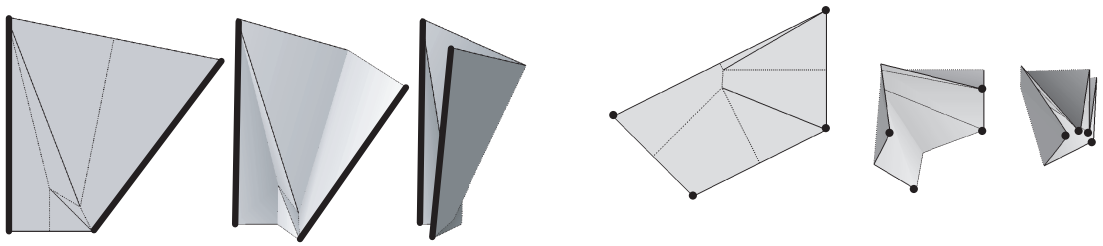


Figure 2:

Edge-tucking molecules are aligned between pairs of edges of the aligned surface polygons. The two edges are moved to a three-dimensionally identical position by folding the edge-tucking molecule between them (Figure 3(a)). Vertex-tucking molecules are aligned between vertices corresponding to one vertex of the three-dimensional surface polygons. The vertices are moved to an identical position by folding the vertex-tucking molecule aligned between them (Figure 3(b)). Hence edge-tucking molecules and vertex-tucking molecules are used for pasting pairs of edges and combinations of vertices respectively.

2.2 Conditions

There are two kinds of conditions for aligning polygons and tucking molecules. One is what we call "2D conditions," which is mainly related to aligning completely unfolded molecules



(a) Edge-tucking molecule (b) Vertex-tucking molecule

Figure 3: (a)An edge-tucking molecule make positions of two segments identical. (b)A vertex-tucking molecule make positions of vertices identical.

on unfolded paper. The other is "3D conditions," which comes from the relations between polygons and molecules when the model is folded. The followings are the conditions for designing polyhedral surface using tucking molecules.

Condition 1 *Polygons and tucking molecules properly tessellate convex planer area.*

Condition 2 *Crease patterns can be generated and do not intersect polygons or each other.*

Condition 3 *The tucks and the polygons do not intersect each other when folded.*

Condition 4 *The tucks have enough angle to get the right curvature of the required surface when folded.*

Condition 1 and 2 are 2D conditions, and Condition 3 and 4 are 3D conditions.

As origami models are designed by placing molecules and generating crease pattern on two dimensional paper, 3D conditions must also be written down as conditions on crease pattern. We use "tuck proxies" to assume three-dimensional positions of the folded tucks, and get sufficient angle inequality conditions on crease pattern.

The procedure and the conditions for proper tessellation are shown in Section 3. The method for generating a crease pattern of the edge-tucking molecule for a given paired edges is shown in Section 4.1 and their conditions in Section 4.2. The method for generating a crease pattern of the vertex-tucking molecule for a given combination of vertices is shown in Section 5.1 and their conditions in Section 5.2. Then 3D conditions are shown in Section 6. Finally, Section 7 shows examples of the application of this method.

3 Tessellation

First, a polyhedral surface is cut so that it is topologically equivalent to a disk if it is not so. Then, edge-tucking molecules and vertex-tucking molecules are added to the surface to split edges and vertices. The order of vertices around the molecules are set according to surface direction.

The polygons and added molecules are mapped onto a convex region of a plane without intersection of the elements or changing of the connectivity. The surface polygons are kept

congruent after mapping, and the tucking molecules are deformed to fit the blank area between the aligned polygons.

The followings are the conditions for properly tessellating planer area by molecules.

Condition 5 *Any mapped tucking molecule does not cross itself.*

Condition 6 *Signed area of any tucking molecule is not negative.*

Condition 7 *The mapped boundary of the surface is convex.*

Figure 4 shows an example of how Condition 5 and 6 help avoid intersections of molecules.

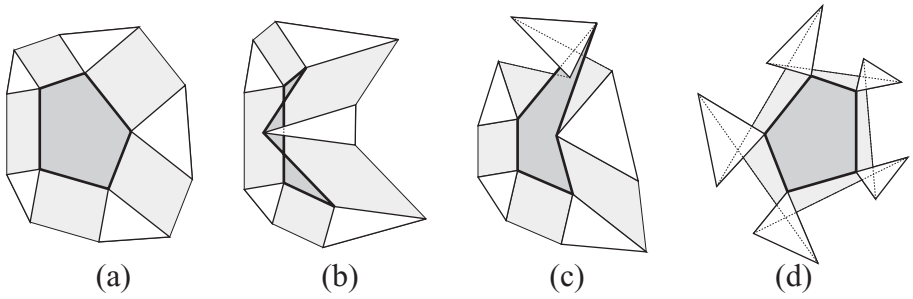


Figure 4: Conditions for properly tessellation. (a) Ok. (b) The vertex-tucking molecule is crossed. (c) The edge-tucking molecule is crossed. (d) The vertex-tucking molecules and edge-tucking molecules have negative area.

4 Edge-Tucking Molecules

4.1 Generating Edge-Tucking Molecule

Figure 5 shows the procedure of generating an edge-tucking molecule. An edge-tucking molecule is a quadrilateral area (AA′B′B) surrounded by the paired edges (AB and A′B′) and segments connecting corresponding vertices of the paired edges (AA′ and BB′).

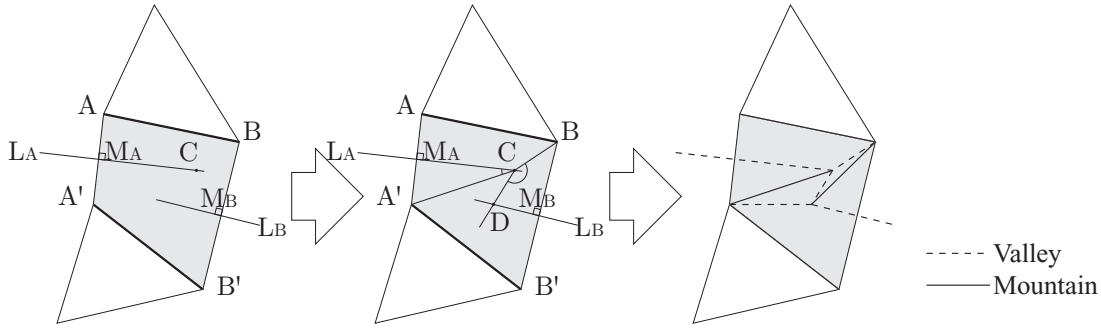


Figure 5: Generating procedure of crease pattern for an edge-tucking molecule. Right image shows the resulting crease pattern.

The basic idea is to make a valley fold along perpendicular bisectors of AA′ and BB′ so that point A is folded to A′ and B to B′, and make a flat-foldable crease pattern that connects these perpendicular bisectors. The followings are the procedure.

1. Define L_A as the perpendicular bisector of segment AA' and define L_B as the perpendicular bisector of segment BB'.
2. Denote the midpoints of segment AA' and of BB' by M_A and M_B respectively.
3. If $AB' = A'B$, then L_A is on L_B and the crease pattern can be get by drawing a valley crease on segment M_AM_B. Else, assume that $A'B < AB'$. (The crease pattern can be constructed in the same way when $A'B > AB'$.)
4. Define point C on L_A inside quad $AA'B'B$.
5. Define point D on L_B so that $\angle M_ACA' + \angle DCB = \pi$
6. Draw crease pattern according to the right image of Figure 5.

In step 5, Kawasaki's theorem [9] is used for flat-foldability of vertex C. Note that Kawasaki's theorem is satisfied also in vertex D. This is always satisfied if you can define point D on L_B.

4.2 Conditions for Generating Edge-Tucking Molecule

The followings are the two conditions for a properly aligned quadrilateral to be an edge-tucking molecule.

Condition 8 *The crease pattern can be generated for each edge-tucking molecule.*

Condition 9 *The generated crease patterns do not intersect each other.*

The necessary and sufficient condition for condition 8 can be written as follows.

Condition 10 *Perpendicular bisectors of* AA' *and* BB' *does not intersect the surface polygons connected to the edge-tucking molecule.*

Figure 6(a) shows the example. This condition is represented as,

$$AB' \geq AB \quad \text{and} \quad A'B \geq AB(= A'B') \tag{1}$$

Instead of Condition 9, we can use the following sufficient condition, which is useful because it can be determined only from the shape of each molecule.

Condition 11 *The generated crease pattern is inside the quadrilateral of the molecule.*

Condition 11 can be represented by angle inequality (Figure 6(b)). Assume that we put point C (a point on L_A from which the new crease line is generated) on M_A, which is the most preferable position to make the crease pattern inside. The condition is satisfied if and only if the point D is inside the quadrilateral $AA'B'B$. This means,

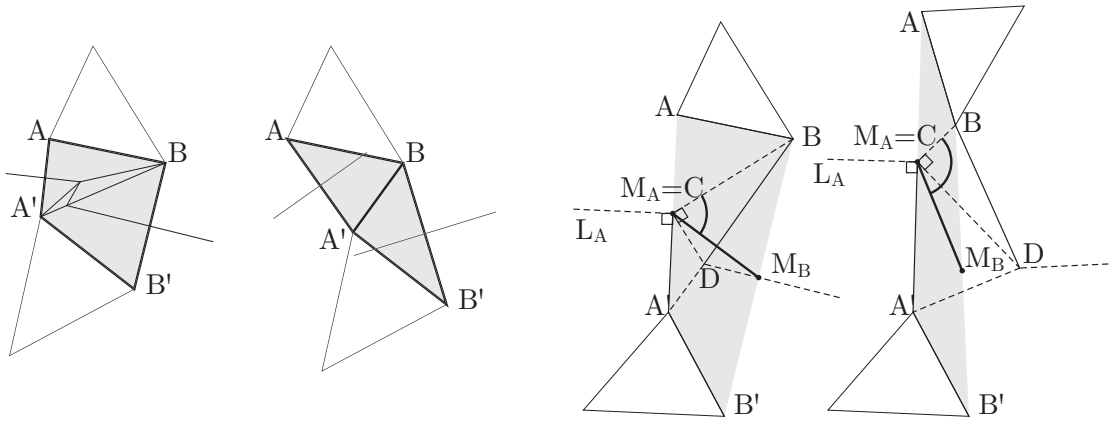
$$\angle BM_AM_B \leq \angle BM_AD \tag{2}$$

Because point D is defined by flat foldability around vertex C,

$$\angle BM_AD = \pi - \angle L_AM_AA = \pi/2 \tag{3}$$

Hence, Condition 11 is represented as,

$$\angle BM_AM_B \leq \pi/2 \tag{4}$$



(a) Left: Ok. Right: A′B < AB

(b) Left: CP is inside. Right: CP is outside

Figure 6: Conditions for an edge-tucking molecule. (a) Conditions to generate crease pattern. (b) Conditions for making the crease pattern inside the molecule.

5 Vertex-Tucking Molecules

5.1 Generating Vertex-Tucking Molecule

Figure 7 shows how the crease pattern for a vertex-tucking molecule is generated. The procedure is as follows.

1. Draw Voronoi diagram with valley crease using the vertices of the molecule as generating points.
2. Connect vertices of each Voronoi polygon to the generating points with mountain crease.
3. Crimp or rabbit ear fold triangles which is not connected to edge-tucking molecules.

The procedure 3 can be omitted in real design. For this is for making the folded shape of the molecule composed of tucks connected to one axis (Figure 10), which is useful for calculating 3D conditions described in Section 6. This kind of molecule based on Voronoi diagram has recently been used for some origami designs. Though the procedure is not precisely described, Robert Lang is showing the same molecule for the connection of strip grafts [4, p.152].

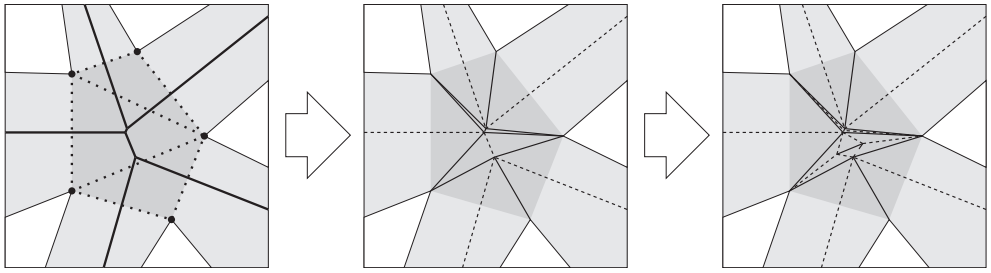


Figure 7: Generating process of crease pattern for a vertex-tucking molecule.

5.2 Conditions for Generating Vertex-Tucking Molecule

The followings are the two conditions for generating vertex-tucking molecules.

Condition 12 *The crease pattern can be generated for each vertex-tucking molecule.*

Condition 13 *The generated crease patterns do not intersect each other.*

Condition 12 is represented as follows.

Condition 14 *The boundary of vertex-tucking molecule is on the edges of Delaunay triangles dual to the generated Voronoi diagram.*

Figure 8(a) shows how this condition works. If a segment on the boundary is not on the Delaunay triangles, an edge-tucking molecule cannot connect to the segment because the perpendicular bisector does not cross the segment.

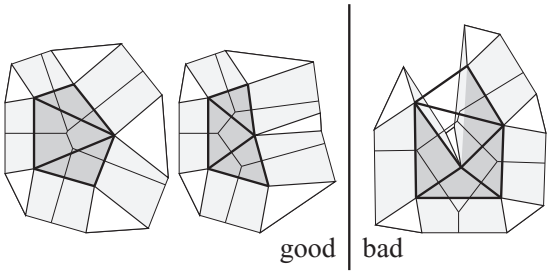
For Condition 13 we use the following sufficient condition instead of the necessary and sufficient condition.

Condition 15 *The generated crease pattern is inside the molecule.*

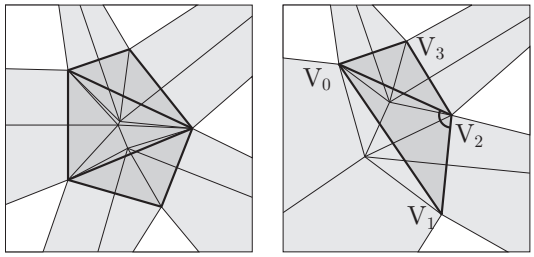
Which is also written as,
For any edge $\mathrm{V_iV_j}$ on the boundary of the molecule and for any vertex $\mathrm{V_k}(k \neq i, j)$ of the molecule,

$$\angle \mathrm{V_jV_kV_i} \leq \pi/2 \tag{5}$$

Note that (5) is also a sufficient condition for Condition 14 as well as for Condition 13. Figure 8(b) shows the example. In the right image, $\angle \mathrm{V_1V_2V_0} > \pi/2$ and the crease pattern get outside the edge $\mathrm{V_1V_0}$.



(a) Left: Ok. Right: The boundary is not on the edges of Delaunay triangles.



(b) Left: CP is inside. Right: CP is outside.

Figure 8: Conditions for a vertex-tucking molecule. (a) Condition to generate crease pattern. (b) Conditions for making the crease pattern inside the vertex-tucking molecule.

6 3D Conditions

3D conditions are conditions on the paper after folded, shown as Condition 3 and 4. Necessary and sufficient conditions for 3D conditions are difficult to be handled by parameters of crease pattern, because the three-dimensional configuration changes not only by the crease pattern, but also by the folding angles of non flat foldable crease lines.

Thus we assume the tucks (i.e., folded tucking molecules) are on predefined "tuck proxies," so that Condition 4 is represented as a condition on parameters of crease pattern (i.e., angles inequalities), while keeping Condition 3.

6.1 Generating Tuck Proxies

We set the widths of tuck proxies as large as possible without intersection with surface polygons or each other, as tuck proxies define the range in which tucking molecules can be folded. Here is an example of generating tuck proxies (Figure 9).

1. Define a segment from each vertex to the direction opposite to the surface direction. The length of the segment is set so that the segment does not intersect the polygons.
2. Connect adjacent segments with strips of two triangles and define them as tuck proxies.
3. If the strips intersect with the surface polygons or each other then shorten the segment length and redo from 1. Else end.

As we assume that folded tucking molecules are on tuck proxies, width of the folded tuck must be less than or equal to width of the tuck proxy. The half of the distance between the corresponding vertices of the tucking molecules can be used for calculating the width of the folded tucks.

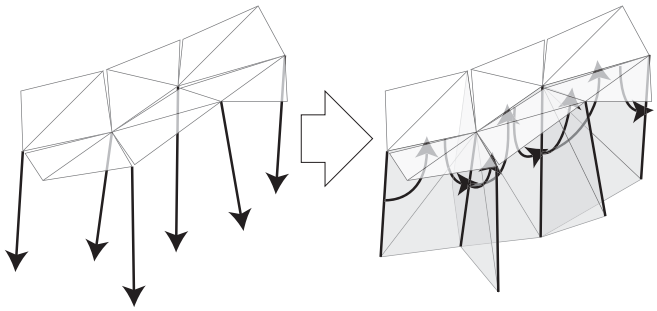


Figure 9: Generating procedure of tuck proxies

6.2 Angle Inequality

Sufficient condition for Condition 4 can be represented by inequalities between angles of folded molecules and angles between edges along tuck proxies.

Relations between the angles on unfolded paper and on folded molecules are shown in Figure 10. In this example $\angle \mathrm{ABB}' < \angle \mathrm{A}'\mathrm{B}'\mathrm{B}$, so $\angle \mathrm{ABB}'$ keeps its angle after folded. General equation is as follows.

$$\min\left(\angle \mathrm{ABB}', \angle \mathrm{A}'\mathrm{B}'\mathrm{B}\right) + \angle \mathrm{O_B BB}'|_{\text{in 2D}} = \angle \mathrm{ABO_B}|_{\text{in 3D}} \tag{6}$$

We name this angle (i.e., $\angle \mathrm{ABO_B}|_{\text{in 3D}}$) "tuck angle" and denote it by $\theta_{i,k}$ for edge i connecting to vertex k.

Folded tucking molecules are easily folded to fit the tuck proxy and crimped to adjust tuck angles of each edge (Figure 11). Note that joint axis (i.e., axis to which the tucks are connected) can be moved by folding the tucks, and one of the tuck angles can be increased. The necessary and sufficient condition for making such adjustment possible for vertex k is as follows
for any edge i and j $(i \neq j)$ connected to vertex k,

$$\theta_{i,k} + \theta_{j,k} \geq \theta'_{i,k} + \theta'_{j,k} \tag{7}$$

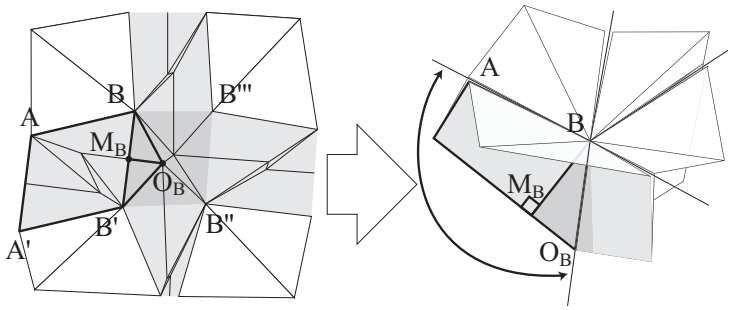


Figure 10: Angle relations between unfolded paper and folded molecules.

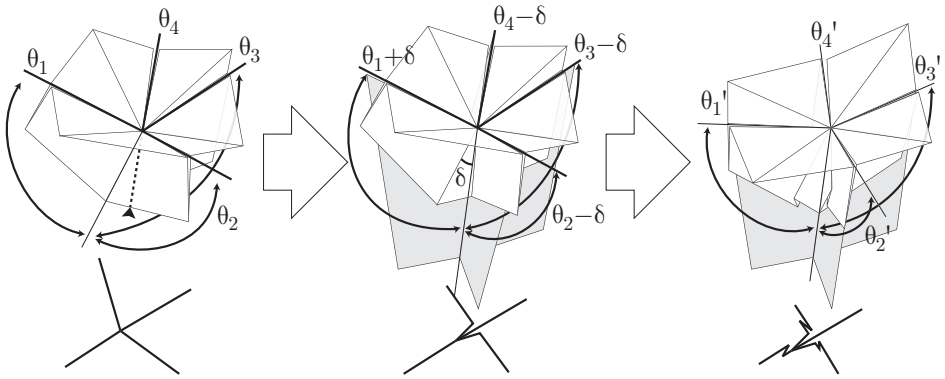


Figure 11: Adjusting tuck angles. (Left to Middle) One of the tuck angles is increased by moving the joint axis. (Middle to Right) Any tuck angles can be reduced by crimping.

where $\theta_{i,k}$ is the original tuck angle calculated by equation (6), and $\theta'_{i,k}$ is the desired tuck angle along tuck proxy.

7 Application

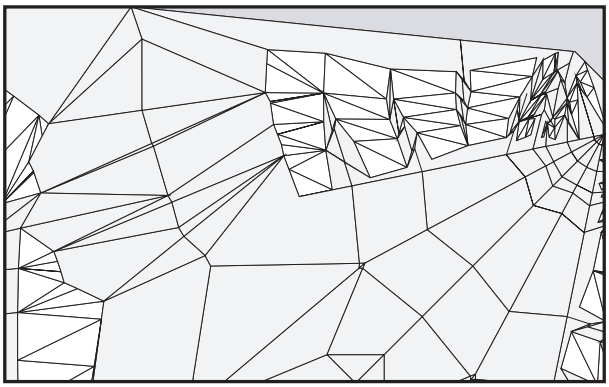
We show two ways of application of the proposed method. Figure 12 shows the examples.

By applying this method for an overall model, a complex, three-dimensionally defined model can be folded by a sheet of paper. Figure 12(a) is a part of the crease pattern for the model "Teapot," whose folded model is shown in the right picture of Figure 1.

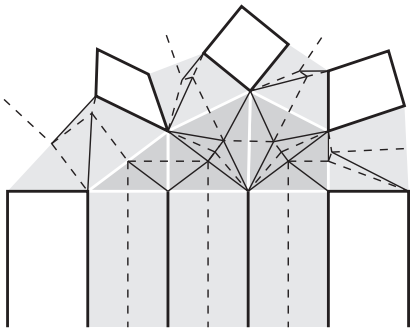
The proposed method is also useful for designing some parts of the model three-dimensionally and connect to other parts designed by other methods. As shown in Figure 12(b), the connection parts can also be flexibly designed using tucking molecules. This approach helps designers to control the degrees of detail and the complexity of the model.

8 Conclusions and Future Works

We proposed a method for designing three-dimensional polyhedral surface using tucking molecules. The generating procedures for building vertex-tucking molecules and edge-tucking molecules were shown. We investigated the conditions for generating the crease pattern (2D conditions), and for constructing desired three-dimensional configuration (3D conditions). We have also shown that the sufficient conditions for 3D conditions can be



(a) Zoom view of the crease pattern for the teapot. See Figure 1 for the whole crease pattern and the folded model.



(b) Connecting 3D design (top) to regular box pleating pattern (bottom).

Figure 12: Examples.

represented by angles and length inequalities on molecules aligned on two-dimensional paper.

The proposed method can be used for designing complicated three-dimensional models and also for adding flexibility to other origami designing methods. The future work is to implement this method and make a tool to help design three-dimensional origamis.

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