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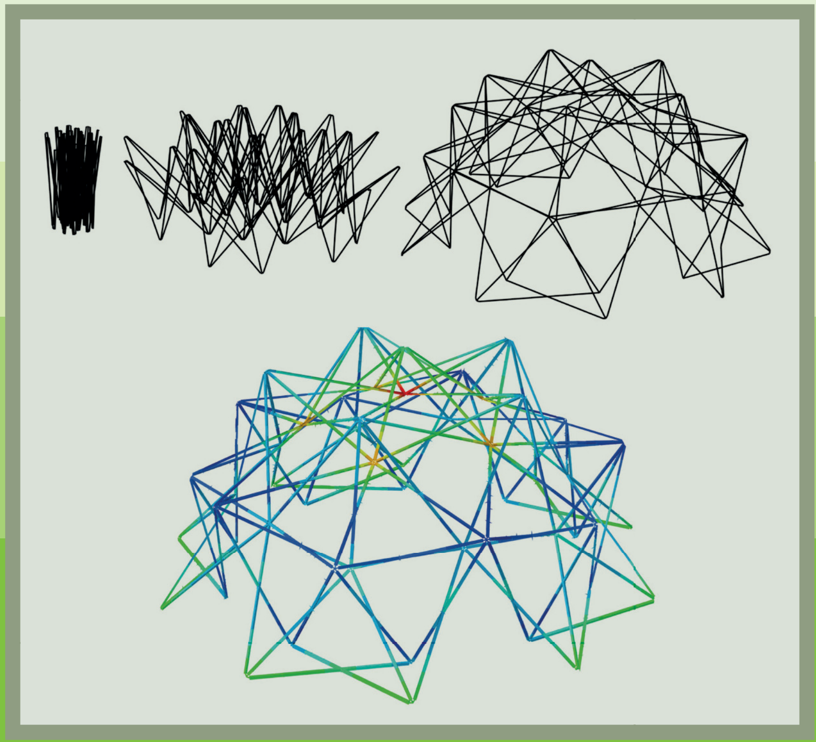

SPECIAL ISSUE

**TRANSFORMABLES: DEPLOYABLE STRUCTURES AND RAPIDLY ASSEMBLED SYSTEMS**

*Guest Editor*: **N. DE TEMMERMAN**

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***COVER:*** *Figures from paper by L. I. W. Arnouts, T. J. Massart, N. De Temmerman and P. Z. Berke*

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# INTRODUCTION TO STRUCTURAL ORIGAMI

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## *ABSTRACT*

*Origami, the art of folding sheets of paper, has been appreciated as a source of inspiration for structural design. Structural origami is a field that is rapidly developing through a collaboration between engineering, design, art, and computation. This paper introduces aspects of structural origami, to share recent exciting developments in this field.*

**Keywords:** *Origami, Structural Origami, Structural Morphology, Folded-plate, Origami Meta-materials*

## 1. INTRODUCTION

Origami, an art form that consists of folding a sheet (or sometimes multiple sheets) of paper into different forms without stretching and often without cutting the sheet, is attracting the interests of not only artists, but also researchers in diverse fields. Folded shapes of origami have been a great source of inspiration for structural designs. The study of *structural origami*, therefore, aims to explore the relationship between structural behavior and the forms that emerge from sheet folding. (Note that the term "origami" comprises of the Japanese word "oru" meaning "to fold" and "kami" meaning "paper," but in structural context, "paper" does not necessarily mean material made with cellulose, but rather means a thin surface with significantly high in-plane stiffness compared to out-of-plane stiffness.)

### 1.1. Background

About a half-century of studies have supported the development of a vibrant academic community, in which artists, mathematicians, scientists, and engineers collaborate to tackle exciting problems that originate from the folding of sheets. One of the earliest meetings on this topic was the *International Association for Shell Structures (IASS) Symposium on Folded Plates and Prismatic Structures* held in Vienna, 1970, which collected research presentations on static, modular, or transformable folded plate systems and shells. Although they were not particularly called "origami," the collected works explored the concepts we now call 'structural origami.' Two of the highlights from the proceedings are the first international appearance of *a new figure composed by the repetition of a "feather pattern"* [1] by Koryo Miura, which we now call *Miura-ori* (Figure 1 left) and a kinematic folded plate system by Resch and Christiansen [2], which we now often call the *Resch Pattern* (Figure 1 right). These patterns are still repeatedly used in scientific origami. The early pioneers explored both the high stiffness and flexibility of sheet folding through repetitive patterns. Such patterns are called *origami tessellations*.

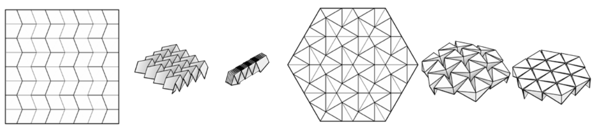

***Figure 1:*** *Miura-ori (Left) and a Resch Pattern (Right)*

A wider interdisciplinary community of scientific origami had started to use the terminology "origami," when the *First International Meeting of Origami Science and Technology* was held in Ferrara, Italy in 1989. The latest of these quadrennial meetings, the *7th International Meeting on Origami Science Mathematics and Education* (7OSME), was held in Oxford, United Kingdom in 2018. The concept of origami is rapidly expanding to include a diverse range of scientific areas, such as material sciences, biology, robotics, and micro electro mechanical systems.

Since the 1990s, computational origami has accelerated interdisciplinary collaboration by providing geometric models, universal algorithms, and accessible software tools. In the IASS, the Structural Morphology Group (SMG or WG 15) has been the platform that supports advances in origami in a structural context. Since 2011, SMG has been holding a study group, named *Origami*, where researchers exploit the geometry of folding through computation and fabrication to find new opportunities for structural engineering. The interdisciplinarity is a key characteristic of this field. New viewpoints and interpretations given from different fields make the topics vibrant.

### 1.2. Fun with Structural Origami

Because of the fluid nature of this field of study, we try to keep the definition of origami flexible. Structural origami deals with any interesting structural study related to surface folding. This paper attempts to share what makes structural origami exciting, rather than provide a systematic review. Therefore, this introduction to structural origami is unstructured.

Section 2 describes the geometric meaning of folding: an out-of-plane deformation accompanying sharp creases. The Origamizer algorithm represents the universality that can be achieved by folding. We mention how natural systems use a combination of folding and growth of surfaces. The nonlinear elastic behaviors of origami go beyond stiffness or flexibility and achieve tunable systems and origami-based mechanical metamaterials. Section 3 briefly shows some approaches for modeling and computing such elastic behavior, or origami. An extreme but scalable model of origami is rigid origami dealt in Section 4; the kinematic study or rigid origami gives design tools for novel transformable structures including overconstrained mechanisms and tubular and cellular structures. In Section 5, we explore the concept of self-folding, a new way of making things that emerge from flat surfaces. The intrinsic difficulty originating from self-folding and its potential use for reprogrammable systems are explained. Section 6 briefly mentions curved folding, which remains unexplored in depth. Section 7 concludes the discussion by introducing our collaborative activities.

## 2. GEOMETRY OF FOLDING

### 2.1. Why Do We Fold?

Paper is a material with a significantly high in-plane stiffness, proportional to the thickness $t$, relative to out-of-plane stiffness, proportional to $t^3$, where $t$ is very small. Theoretically, we consider that the material thickness approaches zero in the limit, so that the surface is completely non-stretching. Such a deformation can be described by an intrinsically isometric mapping of a 2D surface into a 3D Euclidean space. The most important natural behavior of paper is *folding,* defined as out-of-plane deformation that accompanies the formation of creases, i.e., sets of non $C^1$ points along curves.

Paper without creases could be rather boring. To illustrate this, consider a smooth $C^2$ intrinsically isometric mapping, which is well studied in the field of classical differential geometry. *Theorema Egregium* says that the Gaussian curvature, i.e., the product of principal curvatures, is *intrinsic*, i.e., unchanged by intrinsically isometric mapping.

The theorem particularly means that the forms that emerge out of a planar sheet, called a *developable surface,* must have zero Gaussian curvature, meaning that the surface must be singly curved, i.e., locally a cylinder, a cone, or a tangent surface. In other words, doubly curved surfaces like a dome or a saddle require stretching or shrinking of a planar sheet (Figure 2 top). On the other hand, folded planar sheets of paper could form synclastic or anticlastic surfaces as shown in Figure 2, at the bottom. Here, the trick is that the non-smooth creases allow for the surface to wrinkle and thereby, to virtually shrink.

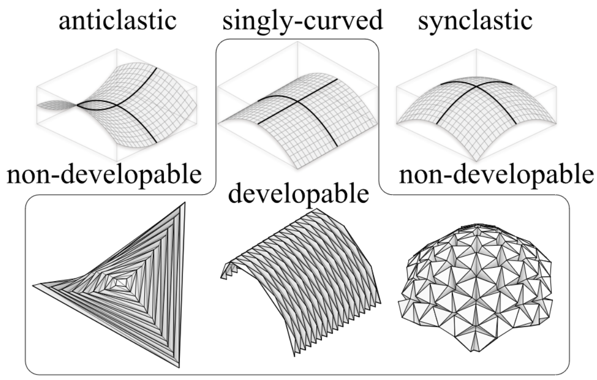


***Figure 2:*** *Top: smooth developable surfaces can form only singly curved surfaces. Bottom: creased surfaces allow for doubly-curved developables (pleated hypar, Yoshimura-pattern, Resch's pattern)*

We may experience such a property of origami, i.e., virtual shrink by creasing, through an easy experiment; take a roll of wax-paper, grab two ends, try to make them shrink. The sheet exhibits self-organized buckling patterns to accommodate the compression without the material itself shrinking (Figure 3). The buckling patterns of paper are characterized by sharp features, i.e., creases, and in the case of a cylinder, a diamond-pattern arises [1]. The shell buckling behavior stands in contrast to the buckling of the one-dimensional elastic rod, forming a smooth elastica curve. Bushnell and Bushnell [3] collect examples and literature of shell buckling coupled with beautiful patterns of creases.

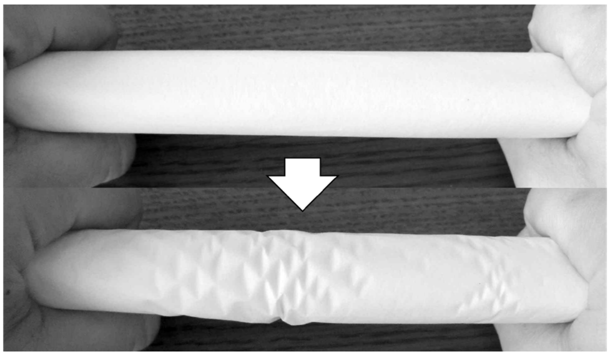

***Figure 3:*** *Buckling patterns of thin sheets are characterized by sharp creases*

## 2.2. Universality of Folding

We have seen that a certain pattern of creases yields a double-curved surface. The inverse of this is called *origami design*, where we ask to obtain the pattern of creases such that the paper transforms into a prescribed form. Computational methods and algorithms are often useful to solve the design problem systematically. The study of origami design was first deeply explored in the 1990s by origami artists and scientists, which brought about the development of so-called super-complex origami.

Now, origami design can be extended into 3D surfaces. Here, the ultimate question is: "Is anything foldable from a 2D sheet of paper?" This problem is numerically solved positively by *Origamizer* [4], software created to generate a crease pattern that folds into an almost arbitrarily given polyhedral surfaces. A constructive proof by Demaine and Tachi [5] showed the universality of origami, i.e., that arbitrarily given polyhedral surfaces can be folded water-tightly.

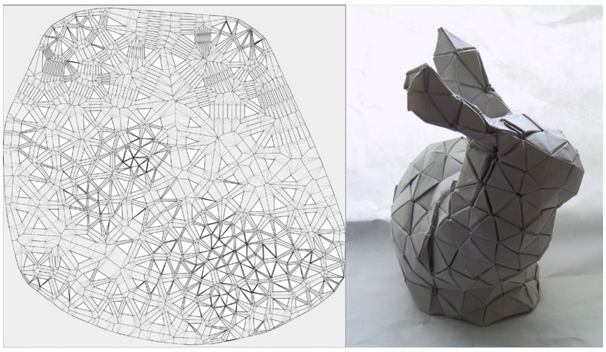

***Figure 4:*** *Crease pattern (left) computed by* Origamizer *and the folded model (right)*

The main idea for the Origamizer is to separately place faces of the polyhedral surfaces leaving some gaps between them. The gaps are folded such that they reconvene, allowing multiple vertices on the paper to fold to a common point, and to allow a pair of edges to fold to a common edge (Figure 5). Such a crease pattern is generated by folding along the Voronoi diagram generated by the placed faces under a certain metric.

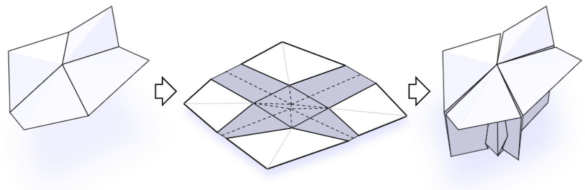

***Figure 5:*** *Folding up to close the gap using a crease pattern based on the Voronoi diagram*

We may interpret this design procedure as the placement of artificial wrinkles on the surface. *Origami tessellations* are a good source of such wrinkles. A freeform surface can be generated out of sheet material by populating an origami tessellation pattern on the surface and solving the developability constraints through optimization [6] (Figure 6).

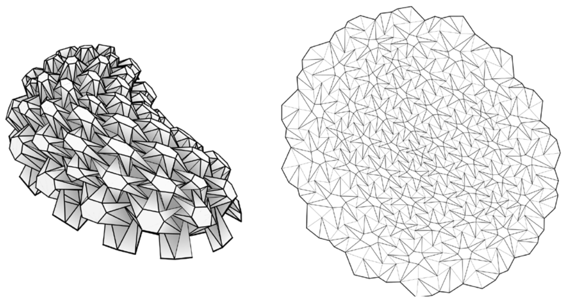

***Figure 6:*** *Freeform origami tessellations by generating Resch-type tessellations on a given surface. Computed using Freeform Origami*

### 2.3. 3D to 3D Folding

The wrinkling idea allows for the approximation of any mapping of a surface to another surface in 3D, if the mapping is contractive, meaning that all distances either shrink, or are kept isometric. This is theoretically underpinned by the Nash-Kuiper theorem [7]. Therefore, we may extend origami beyond developable surfaces. For example, we may start from a curved surface and approximate a flat surface by folding; this is the opposite of making 3D curvature from a flat surface. In this way, we succeed in separating the intrinsic metric of the surface (how it is measured along the surface) and the extrinsic shape (how they look from the outside). Figure 7 illustrates this separation.

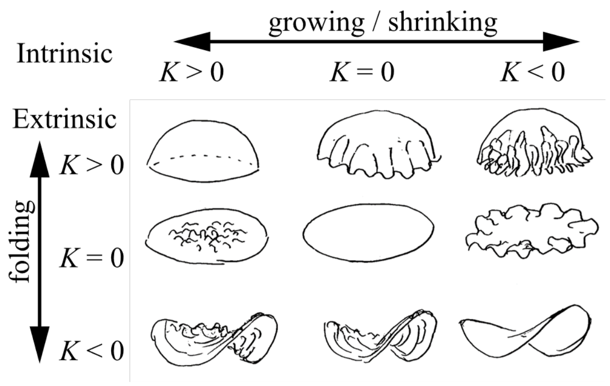


***Figure 7:*** *3D to 3D folding allows for isolation of the intrinsic and extrinsic curvatures. Vertical direction corresponds to folding, while horizontal direction requires growing or shrinking of the material. This contrasts with a smooth ($C^2$) surface along the diagonals, where intrinsic metric and extrinsic curvature correspond*

### 2.4. Growing vs. Folding

The wrinkling technique is also seen in biological systems when biological shells and membranes grow. The surfaces are made by (1) intrinsic deformation where actual material grows through cell division and (2) isometric deformation, i.e., folding, which does not include any intrinsic metric change. The former takes a longer time than the latter process. Clever usage of the coupling of growth with folding is seen in the way a beetle grows their new, bigger shell inside of the older shell. They slowly grow their shells without extrinsic metric change (horizontal motion in Figure 7) with "furrows," and then deploy them isometrically [8].

## 3. STIFFNESS AND FLEXIBILITY

### 3.1. Elastic Behaviors

The extreme difference in the in-plane and out-of-plane stiffnesses of paper leads to varieties of stiff and flexible structures when folded. Stiff origami patterns form folded-plate systems [1] or lightweight sandwich cores [9][10]. Flexible origami patterns may lead to deployable space structures [11], stents [12], or energy absorption devices [13]. Moreover, because of the geometric nonlinearity, some origami structures exhibit multistability [14][15]. Because the behavior depends on the shapes, one may tune structural stability with mono-stable, zero-stiffness, or bi-stable properties [16].

Also, origami tessellations viewed as homogenized materials have unusual behavior; so they are often called *mechanical metamaterials*. In particular, the auxetic property, i.e., property with a negative Poisson ratio, has been explored. Shenk and Guest [10] showed that certain origami tessellations have Poisson ratios of opposite signs for in-plane and out-of-plane deformations; Miura-ori is auxetic against in-plane deformation, i.e., the surface shrinks in both directions, but the Poisson ratio against out-of-plane deformation is positive, i.e., the surface forms an anticlastic surface. An eggbox surface has the opposite property, with positive in-plane and negative out-of-plane Poisson ratios.

In most cases, the exciting elastic behaviors of origami can be well captured via a very simple elastic model of bar-and-hinge when the fold lines are composed only of straight lines. Note that curved creases behave quite differently and present a number of open questions, as we discuss in Section 6.

### 3.2. Bar and Hinge Model

An important principle of folding from differential geometry is that *a straight crease stays straight after folding*. More precisely, folding along a straight crease on a developable surface by any amount except 0 or π will keep the crease straight also in its folded form [17]. In other words, the surfaces need to stretch in order to deform the crease line. So, a straight crease behaves similarly to a stiff bar. Accordingly, a face surrounded by three straight creases is kept stiff and planar. Resch and Christiansen [2] showed the first computational model of kinematic origami, which uses a single triangular finite element for each triangular face, coupled with angular springs representing the hinge stiffness at the creases.

Faces surrounded by more than three creases need more care because they bend in a complex manner accompanying a small amount of stretch [18]. Thus, the behavior is dependent of the thickness of the material. If the surface is extremely thin and thus the in-plane stiffness governs the behavior, a facet gets triangulated with a finite number of sharper creases. When the material gets thicker, the triangulation line starts to avoid sharp creases and gets rounded by compromising the stretch of the material.

A computational model for acquiring such coupled behavior of panel bending has been explored through the *bar-and-hinge* model using a central node (e.g., five nodes and eight bars (N5B8) for a quadrilateral) [19]. The software system *MERLIN* utilizes these bar-and-hinge models to capture the geometrically nonlinear behaviors of elastic origami [20]. The bar-and-hinge model is well contrasted to the diagonal-bar model (e.g., four nodes and six bars (N4B6) for a quadrilateral), which is a straightforward strategy that fails. The diagonal-bar model fails because it produces singular stiffness matrices when panels are planar, but the panel becomes extremely stiff as it gets bent (Figure 8). Please, do not use diagonal bars.

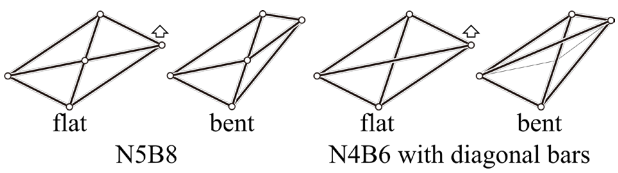


***Figure 8:*** *The N5B8 with hinge stiffness can model consistent bending stiffness. The N4B6 model with diagonal bars can be extremely stiff in its bent state as it forms a tetrahedron*

### 3.3. How to Measure Stiffness?

In the study of structural origami, it often makes more sense not to consider stiffness for a single loading case, but to understand the “stiffness of the structure” if such a measure exists. A very good intuition comes from the eigenvalue analysis. Eigenvalue analysis provides the natural vibration modes of the structure; a high-frequency vibration mode corresponds to high stiffness, and a low-frequency mode corresponds to low stiffness. Schenk and Guest [21] use eigen decomposition of the stiffness matrix to sort the deformation modes from the most flexible to the stiffest. After six trivial rigid body zero-stiffness modes (translation and rotation in 3D), the seventh mode corresponds to the most flexible deformation mode. For a kinematic structure, one may expect the seventh mode to be the desired transformable mode; however, surprisingly, they show that even the Miura-ori sheet, known to be a one-degree-of-freedom (DOF) kinematic origami structure, is more flexible against torsion than folding with some natural set of stiffness parameters. The important fact is that the deformation modes and the eigenvalues relative to each other do not depend on the scale or on material properties, such as the modulus or the density. This is because the properties will only change the eigenvalues proportionally. Therefore, the properties analyzed in this way are intrinsic purely to the shape of the structure.

A further extreme idea also coming from eigenvalue analyses is that of *mode separation*. When considering an ideal transformable structure, we desire high stiffness to produce a light-weight and load-bearing structure on the one hand, and high flexibility to enable transformation, on the other hand. Origami structures used to fail in achieving this contradicting requirement of being stiff yet flexible at the same time. The glide-reflection coupling of origami tubes achieves this best-of-both-world nature [22]. More precisely, the structure exhibits mode separation between the desired folding mode and the undesired warping mode; the desirable deployment mode in the seventh mode is 400 times more flexible than the undesirable deformation in the eighth mode, under plausible assumptions concerning the material thickness and hinge stiffness. Again, this relative property only depends on the shape of the structures and how they are geometrically coupled.

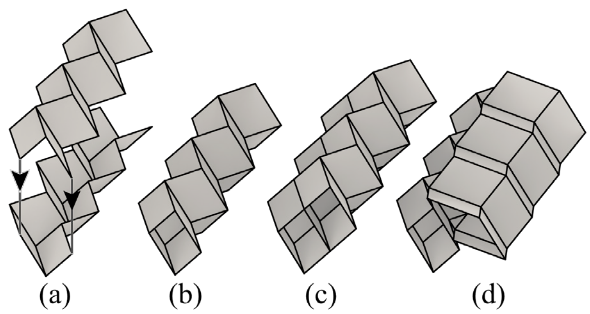


***Figure 9:*** *(a) Miura-ori sheets, (b) origami tube [23], (c) parallel alignment [23], and (d) glide-reflection coupling [22]*

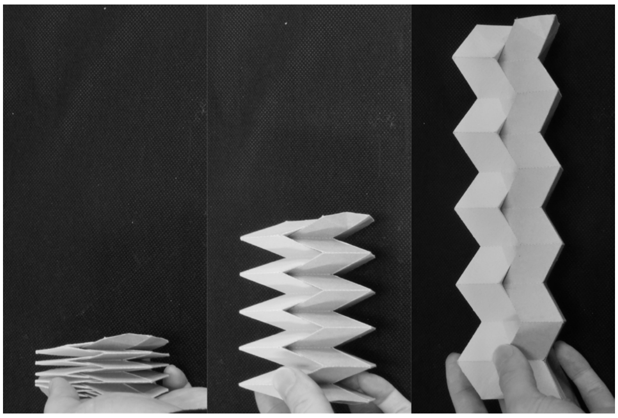

***Figure 10:*** *A glide-reflection coupling of tubes [22] actuated from one end. The stiffness of the 8th mode helps to achieve the uniform, steady deployment*

These computational models of origami help in understanding the elastic behavior of existing structures, but they are a bit indirect for designing structures. In contrast, much inspiration for new origami structures comes from the kinematic theory of *rigid origami* that we explain in the next section. For example, even though the behavior of the glide-reflection coupling of tubes is well-explained by mode separation, the actual design originated from the theory of rigid origami.

## 4. RIGID ORIGAMI

### 4.1. Rigid Origami Model

Rigid origami is a kinematic model of origami where each face is completely rigid and connected by rotational hinges. In other words, the bars and bending hinges have infinite stiffness. We are particularly interested in the existence of a continuous folding motion of rigid origami, called *rigid foldability*. Rigidly foldable mechanisms are scalable. Because the mechanisms do not rely on the material flexibility, they can be realized with panels of arbitrary thicknesses and rotational hinges of different types. Numerous thickness accommodation techniques have thus been explored to utilize rigid origami in large-scale structures [24][25][26].

Theoretical results and design techniques for rigid origami are based on a model parameterized by the fold angles. The *fold angle* of a crease is the signed dihedral angle starting with zero at the unfolded state (usually, a positive value is used for valley crease). The set of fold angles of all creases for a valid folded state is called a *configuration*. Conversely, a set of arbitrary angles does not necessarily represent a valid folded state. This is because most of the interesting origami form a *parallel link mechanism* consisting of closed loops around interior vertices; this constrains the configuration.

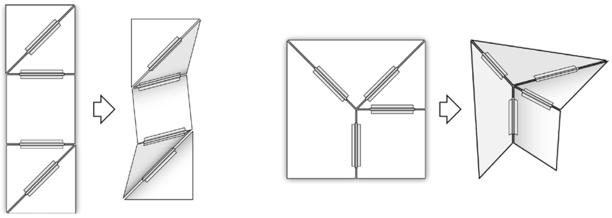

***Figure 11:*** *Left: serial linkage origami without a loop. We can assign arbitrary fold angles to all creases. Right: parallel linkage origami forming a loop around an interior vertex. The fold angles are geometrically constrained*

We may consider a configuration to be a point in an $E$ dimensional space, where the coordinates of the point are the fold angles, and $E$ is the number of creases. In theory, the set of possible configurations, called *configuration space*, can be drawn as a set of the lower (or equal) dimensional manifolds in the $E$ dimensional space. The dimension of the manifold at a regular point is called the *degrees of freedom (DOF)*. Generically, because each vertex produces three equations, the DOF of origami (homeomorphic to a disk) is $E - 3V$, where $V$ is the number of interior vertices.

### 4.2. Overconstrained Origami

However, this computation of DOF is not true for special cases. Special cases are the very source of interesting behaviors in origami. For example, Miura-ori is one of such cases. Figure 12 shows the folding motion of Miura-ori composed of an array of parallelograms. In a generic case, an $m \times n$ array of quadrilaterals would have $E - 3V = -(m - 2)(n - 2) + 1$. This means that the structure would be static if the numbers of rows and columns were both larger than two. However, because of the periodicity of the structure, the actual DOF of Miura-ori is one. Such a system with redundant constraints is called an *overconstrained mechanism*.

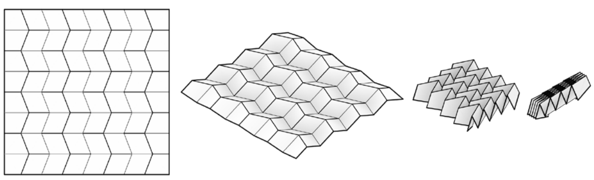

***Figure 12:*** *Miura-ori is an overconsrained mechanism*

Several studies have created overconstrained mechanisms composed of quadrangular quadrivalent mesh by generalizing the Miura-ori structure. Tachi [27][28] showed that the degeneracy of the constraints in Miura-ori is preserved even without the periodicity, leading to a freeform generalization of bidirectionally foldable structures (Figure 13). The design space of such "exotic" quadrilateral mesh is still not fully understood, and novel families have recently been discovered [29][30].

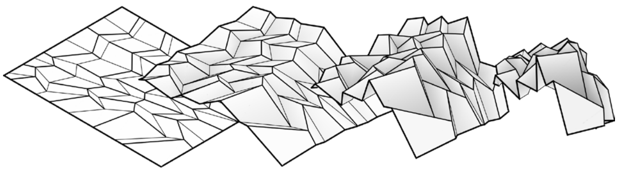

***Figure 13:*** *Freeform generalization of Miura-ori [27]*

### 4.3. Graphical Methods

The constraints of rigid origami around each vertex (assume it is of degree-$k$) are expressed as: the composition of a sequence of rotations about the creases $i = 0, \ldots, k-1$ incident to the vertex equals identity. If crease $i$ has fold angle $\rho_i$ and the crease direction represented by $\mathbf{L}_i$,

$$\prod_{i=0,\ldots,k-1} \mathbf{R}(\mathbf{L}_i, \rho_i) = \mathbf{I},$$

where crease angle $\mathbf{R}(\mathbf{L}_i, \rho_i)$ is the rotation around crease $i$. Taking the derivative of these constraints will give the constraints on the infinitesimal folding motion $\dot{\rho}_i$, or the speed of folding, which is represented as,

$$\sum_{i=0,\ldots,k-1} \mathbf{L}_i \, \dot{\rho}_i = \mathbf{0}.$$

This equation can be interpreted as the equilibrium of forces around the vertex when $\dot{\rho}_i$ is seen as the axial force applied to the crease. This leads to a graphical connection between origami and self-equilibrium structures. Figure 14 shows the same polyhedron seen as an infinitesimally (but not finitely) foldable structure called a *shaky polyhedron* and as the self-equilibrium tensegrity structure [31]. The kinematic shell by Mitchel et al. [32] demonstrates the graphical connection between statics and kinematic structures. Recent studies further explore graphical methods related to second-order infinitesimal folding motion [33][34].

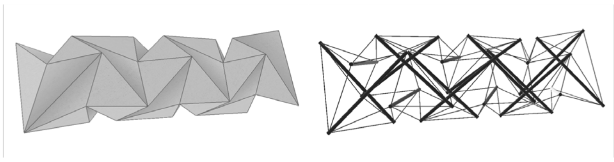

***Figure 14:*** *Shaky polyhedron (left) and tensegrity structure (right)*

### 4.4. Cellular Composition

Further novel structural properties are sought out by going beyond a single sheet. Cellular structures periodically tessellating a volume can be formed by combining sheets. Here, the important constraint for transformable performance is to avoid closed polyhedral packing of a space, e.g., honeycomb sandwich panels. Keeping an open cell is necessary because *Bellows Theorem* states that a closed polyhedron does not rigidly fold while changing its volume [35].

A series of transformable cellular structures can be made, or can be understood, as the stacking of tubular structures. One-DOF rigidly foldable tubes can be made by carefully stacking two folded sheets [23][36]. By choice of the symmetry of the tubes and the stacking, we may obtain different kinds of cellular structures, e.g., cellular structures by translation [23], interleaving [36], and glide-reflection [22]. Stacking of Miura-ori sheets makes a cellular material auxetic in all three dimensions [37] while interleaving structures make a highly anisotropic structure stable in one direction and flexible in two other directions [38] (Figure 15).

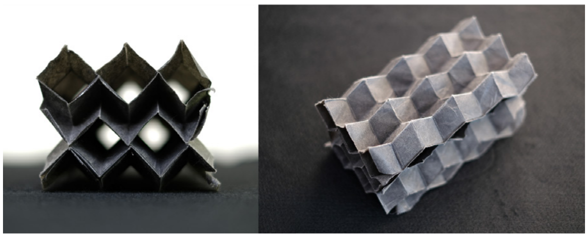

***Figure 15:*** *Interleaved tubular structure [38].*

Another powerful strategy for creating transformable cellular materials is to replace each face of a polyhedron by a cylinder generated by extruding the perimeter of the face; then the cylinder can sheer while keeping the planarity of the face. The technique is known in the origami community as *snapology,* introduced by Heinz Strobl. Applying this replacement technique to a periodic polyhedral packing results in families of multi-DOF cellular structures [39][40].

## 5. SELF-FOLDABILITY

### 5.1. Folding-based Fabrication

In biological systems, forms are fabricated based on self-organization; here, the mechanism of making things is intrinsic to the material. This contrasts with normal artifacts, where the materials and the tools for manufacturing are separately prepared. Such an intrinsic way of making inspires us to develop techniques called *self-folding*, where each crease folds by itself reacting to external energy. Our goal is to find a proper patterning creases that can selectively actuate hinges such that the entire structure folds up into the desired shape.

Several methods realize self-folding through gradational material shrinking and expansion patterned on a crease, resulting in a controlled bending moment. The crease can be patterned through three layers of materials, where some of the layers are designed to shrink or swell (Figure 16). Na et al. [41] use swelling polymer gels to induce the folding and unfolding of origami tessellations in microstructures. Felton et al. [42] use the one-time fabrication method of a robot using heat shrink plastic sheets.

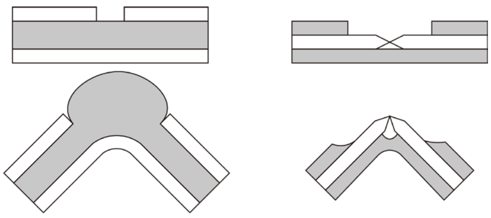

***Figure 16:*** *Swelling the middle layer[41](left) or shrinking the outer layer[42](right) leads to self-folding*

By combining the self-folding techniques and the universality of origami, we may dream of a future where we can fabricate desired functional shapes by just printing a pattern and heating it in an oven. Of course, there are several issues that need to be overcome to achieve this goal. An obvious improvement is to have more precise control over the magnitude and the timing of the applied bending moment. However, there is a typical issue intrinsic to origami that cannot be solved even if the applied force is fully controllable, namely, the bifurcation problem.

### 5.2. Bifurcation

When folding a sheet of paper manually, it is more difficult to fold a 3D shape from a 2D sheet of paper than it is to unfold a 3D shape into a 2D form. Folding requires a proper "sequence of folding," otherwise the paper folds into other things or modes. Once the structure takes a path into a different mode, the only way to correct is to go back to a flat shape (Figure 17).

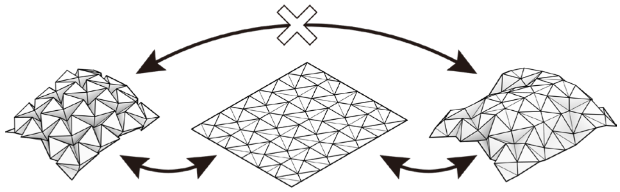

***Figure 17:*** *A mistake in the folding (right) can be fixed (left) only through returning to the entirely unfolded state (middle)*

This irreversibility is also true even for one-DOF rigidly foldable origami with fully controlled self-folding actuation. To illustrate the behavior precisely, Figure 18 shows the configuration space of a degree-4 vertex. Because a degree-4 vertex is a one-DOF structure, the configuration space is drawn as a set of curves in 4D space. Specifically, two curves meet at one point; the meeting point is the flat unfolded state. At the flat configuration, the configuration space bifurcates into two paths, each of which corresponds to a different folding mode.

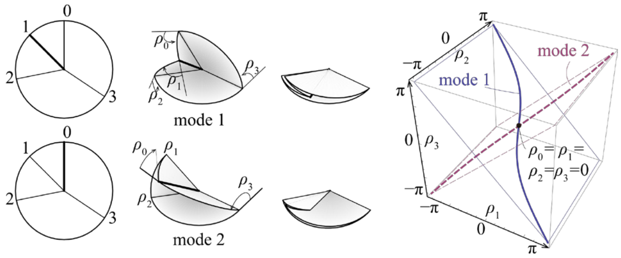


***Figure 18:*** *The configuration space of a degree-4 vertex (right) and its corresponding folding modes (left)*

*Self-foldability* [43] is the problem of judging if there is a pattern of applied force that consistently pushes the configuration into the correct branch so that it folds to a specified folded state. In the case of the degree-4 vertex, there is an assignment of bending moment that ensures the potential energy decreases only along the correct path.

Now consider a multi-vertex system. For each vertex, there is a binary decision of choosing a mode. Therefore, we may potentially get an exponential number of folding modes through the combination of each bit, while some of the mode assignments are restricted in an overconstrained mechanism. Combinatoric analyses show that the number of combinations can rapidly explode [44]. (This combinatoric behavior also makes the rigid foldability problem of a given origami pattern computationally hard, specifically, NP-hard [41].) Self-folding fails when the dimension of the control is less than the number of modes. However, the dimension of the control increases only, at most, linearly with the number creases, while the number of modes can increase exponentially. So, there is a general tendency that the more complex the pattern, the more likely the self-folding fails. The probability of success will approach zero just by making the pattern complex.

Kinematic bifurcation, as explained above, is an annoying problem on one hand. However, it can be potentially exciting, because a single sheet can fold into different separate mechanisms. For example, we may obtain re-programmable metamaterials that can switch between stiff and flexible, auxetic and normal, or multi-DOF and one-DOF. The author and the collaborators are now working on several approaches to tame this bifurcation (Figure 19).

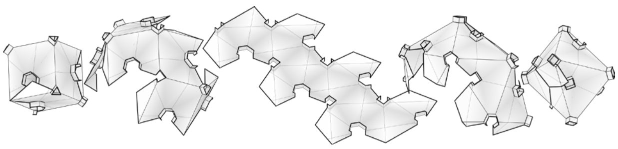

***Figure 19:*** *A structure that folds into a cube and an octahedron [46]*

## 6. CURVED FOLDS

The ideas explained above come from an assumption that a creased line is a straight line. If we use a curved crease, the surface exhibits the coupled behavior of crease folding and surface bending. Curved folding is an extremely rich source of designs. Artists and designers have explored varieties of forms using curved folding, and there have been attempts at computational analysis and designs [47].

However, these theories are still under development, when compared to straight-line origami. The typical difficulty in the geometry of curved folding is in finding a proper model that can capture both the discrete and continuous nature of creases and surfaces. An approach using differential geometry was recently updated to represent the constraints between curved creases [48]. This succeeds in reconstructing some of David Huffman's curved folding sculptures. However, the majority of existing curved folded forms are not yet mathematically described.

The elastic behaviors of curve-folded surfaces are even more exciting but are even less understood. Future studies that bridge the geometric study of curved folding and the theory of active bending could aid in understanding such systems.

## 7. CONCLUSION: LET'S WORK TOGETHER

The structural origami group (Rupert Maleczek and the author) initiated annual collaborative workshops, called "Structural Origami Gatherings," where architects, engineers, mathematicians, and computer scientists collaboratively solved shared questions on structural origami. (This style is borrowed from computer science community. Erik Demaine calls the style *supercollaboration.*) The problems range from kinematic design, self-folding actuation control, elastic behavior analysis, structural systems for freeform surfaces, bending active curved folding systems, etc.

We believe that structural origami is not a subtopic closed within engineering, but rather, it looks like a cloud of academic interests, floating over art, architecture, engineering, science, informatics, and math. It turns out that collaboration is helpful especially because it provides new, unexpected perspectives on the problem, leading to a dozen new questions.

This collaborative nature is not specific to origami but is universal in the field of shells and spatial structures. We believe that SMG in IASS works as the platform for cultivating interdisciplinary discussions and collaborations between specialists from different fields such as engineers, architects, artists, mathematicians, biologists, using the shared language of "forms." The SMG Workshop during the IASS Symposium in Boston, in 2018, was started as a way to bridge and reunite structural studies on origami, scissors, bending active structures, and graphical statics.

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